§5.5. Substitution Rule (U-Sub)

Recall the notation differentials in §3.10 (2A)

$u = u(x)$, then $\boxed{du = u'(x)\,dx}$

e.g. $u = \sin\theta$, $du = \cos\theta \cdot d\theta$

$u = 1 + x^2$, $du = 2x \cdot dx$

U-sub: A method to (change) (simplify) the expression of definite integral via CHANGING INTEGRATION VARIABLE.

Formula: If $u = g(x)$, then

$$\int \underbrace{f(g(x)) \cdot g'(x)\,dx}_{\text{ugly and complicated}} = \int \underbrace{f(u) \cdot du}_{\text{neat and clean}}.$$

Three Types (Basic Types) + Two special sub.s

① $\int e^{5x}\,dx$, $\int \sqrt{2x+1}\,dx$.

② $\int 2x \cdot \sqrt{x^2+1}\,dx$ and $\int x^3 \cdot \cos(x^4-2)\,dx$.

③ $\int x^5 \sqrt{x^2-2} \cdot dx$.

④ $\int \tan x \, dx$. ← (trig-sub)

⑤ $\int \frac{3(\ln x)^2}{x} dx$ ← (ln-sub)

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①. $\int e^{5x} dx$.

Step 1: Set $u = 5x$.

Step 2: differential: $du = 5 dx$.

Step 3: Solve for dx: $dx = \frac{du}{5}$

$\boxed{\text{Step 4: Plug in:}}$ (Replace all "x-terms" via "u-terms")

$$\int e^{5x} \cdot dx = \int e^u \cdot \frac{du}{5} = \int \frac{1}{5} \cdot e^u \cdot du$$

Step 5: integrate (with respect to u)

$$= \frac{1}{5} \cdot e^u + C.$$

Step 6: Back to x.
(from u).

$$= \boxed{\frac{1}{5} \cdot e^{5x} + C.}$$

$\int \sqrt{2x+1} \, dx$.

$\boxed{u = 2x+1, \quad du = 2 \cdot dx \quad dx = \frac{du}{2}.}$

Plug in

$$= \int \sqrt{u} \cdot \frac{du}{2}$$

$$= \int \frac{1}{2} \cdot u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}+1} \cdot u^{\frac{1}{2}+1} + C = \frac{1}{3} u^{\frac{3}{2}} + C = \boxed{\frac{1}{3}(2x+1)^{\frac{3}{2}} + C.}$$

② . U-sub: "U" stands for "Ugly".

$$\int 2x \cdot \sqrt{x^2+1}\, dx$$

Set up $u = x^2+1$ ← "ugly part"

$du = 2x \cdot dx$ differentiate

solve for $dx = \dfrac{du}{2x}$.

Plug in

$$= \int \cancel{2x} \cdot \sqrt{u} \cdot \frac{du}{\cancel{2x}}$$

$$= \int \sqrt{u} \cdot du$$

$$= \frac{1}{\frac{1}{2}+1} \cdot u^{\frac{1}{2}+1} + C = \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{3} \cdot (x^2+1)^{\frac{3}{2}} + C}$$

$$\int x^3 \cdot \cos(x^4-2)\, dx$$

Set up $u = x^4-2$,

$du = 4x^3\, dx$

$\Rightarrow \dfrac{du}{4x^3} = dx$.

Plug in

$$= \int x^3 \cdot \cos(u) \cdot \frac{du}{4x^3}$$

$$= \int \frac{1}{4} \cdot \cos u \cdot du$$

$$= \frac{1}{4} \cdot \sin u + C$$

$$= \boxed{\frac{1}{4} \sin(x^4-2) + C}$$

☆☆ ③ $\int x^5 \cdot \sqrt{x^2-2}\, dx$.

$u = x^2 - 2$

$du = 2x \cdot dx$.

$\frac{du}{2x} = dx$

$= \int x^5 \cdot \sqrt{u} \cdot \frac{du}{2x}$.

(now new problems appears: x^5 and $2x$ cannot be canceled out

but still, we can kill something. So, simplify first)

$= \int \frac{x^4}{2} \cdot \sqrt{u} \cdot du$

there is a "x^4" left, try to substitute this part by u.

$= \int \frac{(u+2)^2}{2} \cdot \sqrt{u} \cdot du$

via $u = x^2 - 2$

solve for x^2, $x^2 = u+2$

$\downarrow$

$\underline{x^4 = (u+2)^2}$

Plug in

$= \int \frac{1}{2} \cdot (u^2 + 4u + 4) \cdot u^{\frac{1}{2}} du$

$= \int \frac{1}{2} \cdot u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + 2u^{\frac{1}{2}} du$.

$= \frac{1}{2} \cdot \frac{1}{\frac{5}{2}+1} u^{\frac{5}{2}+1} + 2 \cdot \frac{1}{\frac{3}{2}+1} \cdot u^{\frac{3}{2}+1} + 2 \cdot \frac{1}{\frac{1}{2}+1} \cdot u^{\frac{1}{2}+1} + C$

$= \frac{1}{7} \cdot u^{\frac{7}{2}} + \frac{4}{5} \cdot u^{\frac{5}{2}} + \frac{4}{3} \cdot u^{\frac{3}{2}} + C$

$= \frac{1}{7} (x^2-2)^{\frac{7}{2}} + \frac{4}{5} \cdot (x^2-2)^{\frac{5}{2}} + \frac{4}{3} \cdot (x^2-2)^{\frac{3}{2}} + C$.

(4) $\int \tan x \cdot dx$. do trig-algebra first.

$\tan x = \frac{\sin x}{\cos x}$.

$= \int \frac{\sin x}{\cos x} \cdot dx$.

$u = \cos x$ (Q: why not $u = \sin x$)

$du = -\sin x \cdot dx$.

$-\frac{du}{\sin x} = dx$.

$= \int \frac{\sin x}{u} \cdot \frac{du}{-\sin x}$

$= \int -\frac{1}{u} du$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$.

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(5) $\int \frac{3(\ln x)^2}{x} dx$. Pay attention to the combination of $\ln x$ and $\frac{1}{x}$, since $(\ln x)' = \frac{1}{x}$

So $u = \ln x$, then $du = \frac{1}{x} dx$.

$= \int 3u^2 \cdot du$

$= u^3 + C$

$= (\ln x)^3 + C$.

(5.1) $\int_1^2 \frac{3(\ln x)^2}{x} dx = (\ln x)^3 \Big|_1^2$

$= (\ln 2)^3 - (\ln 1)^3 = (\ln 2)^3$.

Or $\int_1^2 \frac{3(\ln x)^2}{x} dx = \int_{\ln 1}^{\ln 2} 3u^2 du = u^3 \Big|_{\ln 1}^{\ln 2} = (\ln 2)^3$

e.g. $\int_0^1 x \cdot e^{-x^2} dx$. $\quad u = -x^2$

then $u(0) = 0$, $u(1) = -1$

$du = -2x \cdot dx$

$\frac{du}{-2x} = dx$.

$$= \int x \cdot e^u \frac{du}{-2x}$$

$$= \int_{u(0)}^{u(1)} -\frac{1}{2} \cdot e^u \cdot du$$

the change of variable $u = -x^2$ send the lower and upper limit $x=0$, $x=1$ to the new limit $u(0)=0$ and $u(1)=-1$

$$= \int_0^{-1} -\frac{1}{2} \cdot e^u du$$

$$= -\frac{1}{2} \cdot e^u \Big|_0^{-1}$$

$$= -\frac{1}{2} \cdot e^{-1} - \left(-\frac{1}{2} \cdot e^0\right)$$

$$= \boxed{-\frac{1}{2} \cdot e^{-1} + \frac{1}{2}}.$$

Alternative approach: Same u-sub

$u = -x^2$, $du = -2x\,dx$

$\frac{du}{-2x} = dx$.

$$\int_0^1 x \cdot e^{-x^2} dx$$

$$= \int_0 x \cdot e^u \cdot \frac{du}{-2x}$$

$$= \int \frac{e^u}{-2} du = \frac{e^u}{-2} = \frac{e^{-x^2}}{-2} \Big|_0^1$$

$$= \frac{e^{-1}}{-2} - \frac{e^{-0}}{-2} = \boxed{-\frac{e^{-1}}{2} + \frac{1}{2}}$$

substitute the linear function inside Trig-function.

e.g. 2) c). in Sample Midterm 1. $\int \sin(7\theta+5)\,d\theta$, $u=7\theta+5$.

(§5.5).

12 $\int \sec^2 2\theta\, d\theta$, $u=2\theta$, $du=2\cdot d\theta$

53. $\int_0^1 \cos^2\left(\frac{\pi t}{2}\right)dt$, $u=\frac{\pi t}{2}$, $du=\frac{\pi}{2}\,dt$

☆ 5

57. $\int_0^{\pi} \sec^2(t/4)\,dt$, $u=t/4$, $du=dt/4$.

☆☆ 72 $\int_0^{\frac{T}{2}} \sin(2\pi t/T-\alpha)\,dt$, $u=2\pi t/T-\alpha$, $du=\frac{2\pi}{T}\cdot dt$

T, α are constants.

~~Remark: I will not put trigonometric substitution in Midterm 1.~~

~~see 7th Ed. §5.5, 5, [illegible], 40, 39, 41.~~

~~where the functions are all products of Trigonometric functions~~

~~(will appear in Chapter 7).~~

Substitute with inverse trig-function

$u=\tan^{-1}x$, $u=\sin^{-1}x$ will appear

☆ substitute the function $\boxed{\text{inside}}$ Trig. function.

☆ ⑦

6. $\int \frac{\sec^2(\frac{1}{x})}{x^2}\,dx$, $\quad u=\frac{1}{x}$, $\quad du=-\frac{1}{x^2}dx$

7. $\int x\cdot\sin(x^2)\,dx$, $\quad u=x^2$, $\quad du=2\cdot x\,dx$.

16. $\int e^x\cdot\cos(e^x)\,dx$. $\quad u=e^x$, $\quad du=e^x\cdot dx$.

18. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. $\quad u=\sqrt{x}$, $\quad du=\frac{1}{2}\cdot x^{-\frac{1}{2}}dx$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot dx.$$

24. $\int\sqrt{x}\cdot\sin(1+x^{\frac{3}{2}})\,dx$. $\quad u=1+x^{\frac{3}{2}}$, $\quad du=\frac{3}{2}\cdot x^{\frac{1}{2}}dx$

$$=\frac{3}{2}\cdot\sqrt{x}\cdot dx$$

32. $\int\frac{\sin(\ln x)}{x}\,dx$, $\quad u=\ln x$, $\quad du=\frac{1}{x}dx$.

34. $\int\frac{\cos(\pi/x)}{x^2}\,dx$, $\quad u=\frac{\pi}{x}$, $\quad du=\pi(-\frac{1}{x^2})dx=-\pi\cdot\frac{1}{x^2}dx$

42. $\int\sin t\cdot\sec^2(\cos t)\,dt$, $\quad u=\cos t$, $\quad du=-\sin t\cdot dt$

29. $\int 5^t\sin 5^t\,dt$, $\quad u=5^t$, $\quad du=\ln 5\cdot\underline{5^t\,dt}$

62. $\int_0^{\frac{\pi}{2}}\cos x\cdot\sin(\sin x)\,dx$, $\quad u=\sin x$, $\quad du=\cos x\cdot dx$.

sln to 62.
$$\int_0^{\frac{\pi}{2}}\cos x\cdot\sin(\sin x)\,dx=\int_0^{\frac{\pi}{2}}\sin(\sin x)\cos x\,dx$$
$$=\int\sin u\cdot du$$
$$=-\cos u$$
$$=-\cos(\sin x)\Big|_0^{\frac{\pi}{2}}=-\cos(\sin\tfrac{\pi}{2})-(-\cos(\sin 0))$$
$$=-\cos 1+\cos 0=-\cos 1+1$$

☆ substitution with Inverse Trig function

(Rmk: whenever you see inverse trig. functions, set u equal to them)

☆ ⑧

30. $\int \frac{\tan^{-1}x}{1+x^2}dx$, $u=\tan^{-1}x$, $du=\frac{1}{1+x^2}dx$

43. $\int \frac{dx}{\sqrt{1-x^2}\cdot\sin^{-1}x}$ $u=\sin^{-1}x$, $du=\frac{1}{\sqrt{1-x^2}}dx$.

70. $\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$, $u=\sin^{-1}x$, $du=\frac{1}{\sqrt{1-x^2}}dx$.

s/n to 70: $\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int_{\sin 0}^{\sin^{-1}\frac{1}{2}} u\cdot du = \int_0^{\frac{\pi}{6}} u\,du$

$= \frac{1}{2}u^2\Big|_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\cdot(\frac{\pi}{6})^2 - \frac{1}{2}\cdot 0$

$= \frac{\pi^2}{72}$ ☆

☆. Alway substitute $\boxed{\ln x}$

☆ ⑨

21. $\int_1 \frac{(\ln x)^2}{x}dx$

32. $\int \frac{\sin(\ln x)}{x}dx$

69. $\int_e^{e^4} \frac{dx}{x\cdot\sqrt{\ln x}}$

$\Big\}$ $u=\ln x$, $du=\frac{1}{x}dx$.

s/n to 69. $= \int_e^{e^4} \frac{1}{\sqrt{\ln x}}\cdot\frac{1}{x}dx = \int_{\ln e}^{\ln e^4} \frac{1}{\sqrt{u}}\cdot du = \int_1^4 u^{-\frac{1}{2}}du$

$= 2u^{\frac{1}{2}}\Big|_1^4 = 2\cdot 4^{\frac{1}{2}} - 2\cdot 1^{\frac{1}{2}}$

$= 2$

substitution relate to exponential function.

☆

31 $\int e^{\tan x}\cdot \sec^2 x\, dx$, $u=\tan x$, $du=\sec^2 x\cdot dx$

28 $\int e^{\cos t}\cdot \sin t\, dt$. $u=\cos t$, $du=-\sin t\, dt$

☆

8 $\int x^2 e^{x^3} dx$ $u=x^3$, $du=3\cdot x^2 dx$

59. $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$, $u=\frac{1}{x}$, $du=-\frac{1}{x^2}dx$

60. $\int_0^1 x\cdot e^{-x^2} dx$ $u=-x^2$, $du=-2x\,dx$

sln to 60: $=\int_{u(0)}^{u(1)} e^u\cdot \frac{du}{-2} = \int_0^{-1} \frac{1}{-2}\cdot e^u du$

$= \frac{1}{-2}\cdot e^u \Big|_0^{-1} = \frac{1}{-2}\cdot e^{-1} - \frac{1}{-2}\cdot e^0$

$= -\frac{1}{2}e^{-1}+\frac{1}{2}$ ☆

☆

16 $\int e^x \cos(e^x)\, dx$ $u=e^x$, $du=e^x du$

7 $\int \frac{e^u}{(1-e^u)^2} du$, $v=1-e^u$, $dv=-e^u du$

25 $\int e^x \sqrt{1+e^x}\, dx$ $u=1+e^x$, $du=e^x dx$

71. $\int_0^1 \frac{e^z+1}{e^z+z} dz$, $u=e^z+z$, $du=(e^z+1)dz$.

sln to 7 $\int \frac{e^u}{(1-e^u)^2} du = \int \frac{1}{(1-e^u)^2}\cdot e^u du$

$= \int \frac{1}{v^2}\cdot(-dv) = -\int v^{-2} dv$

$= -(-v^{-1})+C$

$= (1-e^u)^{-1}+C$.

Substitution Rule

Classification of Problems (Ex in 7th Edition)

☆ substitute the "thing" under (squre) root.

2). a). in Sample midterm. $\int x\sqrt{3x^2-1}\,dx$.

§5.5.

3. $\int x^2\sqrt{x^3+1}\,dx$, $u=x^3+1$. $du=3\cdot x^2\,dx$

11. $\int x\sqrt{3x^2-1}\,dx$, $u=3x^2-1$, $du=6\cdot x\,dx$

14. $\int u\sqrt{1-u^2}\,du$, $v=1-u^2$, $dv=-2u\,du$

☆ ①

55. $\int_0^1 \sqrt[3]{1+7x}\,dx$, $u=1+7x$, $du=7\cdot dx$

64. $\int_0^a x\sqrt{a^2-x^2}\,dx$, $u=a^2-x^2$, $du=-2x\,dx$

65. $\int_0^a x\sqrt{x^2+a^2}\,dx$, $u=x^2+a^2$, $du=2x\,dx$

☆☆ ①'

19. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$, $u=3ax+bx^3$, $du=(3a+3bx^2)dx$
$=3(a+bx^2)dx$

63. $\int_0^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}}$. $u=1+2x$. $du=2\cdot dx$.

s/n to 63: $\int_0^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}}=\int_{u(0)}^{u(13)}\frac{1}{\sqrt[3]{u^2}}\cdot\frac{du}{2}=\frac{1}{2}\int_1^{27}u^{-\frac{2}{3}}\,du$

$=\frac{1}{2}\cdot 3\cdot u^{\frac{1}{3}}\Big|_1^{27}$

$=\frac{3}{2}\cdot 27^{\frac{1}{3}}-\frac{3}{2}\cdot 1^{\frac{1}{3}}=3$

② §5.5. 25. $\int e^x \cdot \sqrt{1+e^x}\,dx$, $u = 1+e^x$, $du = e^x dx$

$= \int \sqrt{u} \cdot du$

$= \frac{2}{3} u^{\frac{3}{2}} + C$

$= \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C$

69. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$, $u = \ln x$, $du = \frac{1}{x}dx$

$= \int_1^4 \frac{du}{\sqrt{u}}$ $\quad x = e$, $u = \ln e = 1$; $x = e^4$, $u = \ln e^4 = 4$

$= \frac{1}{-\frac{1}{2}+1} \cdot u^{-\frac{1}{2}+1} \Big|_1^4 = 2u^{\frac{1}{2}} \Big|_1^4 = 2\sqrt{4} - 2 \cdot 1 = 2$

☆ Substitute the "Ugly Part" under the "Power"

2 $\int x^3(2+x^4)^5 dx$ $\quad u = 2+x^4$ $\quad du = 4x^3 dx$

9 $\int (1-2x)^9 dx$ $\quad u = 1-2x$ $\quad du = -2dx$

☆ ③ 27 $\int (x^2+1)(x^3+3x)^4 dx$, $\quad u = x^3+3x$, $\quad du = (3x^2+3)dx = 3(x^2+1)dx$

10 $\int (3t+2)^{2.4} dt$ $\quad u = 3t+2$ $\quad du = 3dt$

54 $\int (3t-1)^{50} dt$ $\quad u = 3t-1$ $\quad du = 3dt$

substitute the thing in the "denominator"
(which is actually a "negative" power).

☆ ③
- 4. $\int \frac{dt}{(1-6t)^4}$, $u=1-6t$, $du=-6\cdot dt$.
- 13. $\int \frac{dx}{5-3x}$, $u=5-3x$, $du=-3\cdot dx$.
- 26. $\int \frac{dx}{ax+b}$, $u=ax+b$, $du=a\cdot dx$.
- 56. $\int_0^3 \frac{dx}{5x+1}$, $u=5x+1$, $du=5\,dx$.
- 20 $\int \frac{z^2}{z^3+1}dz$, $u=z^3+1$, $du=3\cdot z^2 dz$.

☆☆☆ following problems still have the "root" or "power" part, but extra transformation needed. (partial s/n provided here, complete s/n can be found in email; Oct. 3/s...

☆☆☆ ④

46. $\int x^2\sqrt{2+x}\cdot dx$.

s/n: $u=2+x$, $\boxed{\Rightarrow x=u-2}$, $du=dx$

$$\int x^2\sqrt{2+x}\,dx=\int (u-2)^2\cdot\sqrt{u}\cdot du$$
$$=\int (u^2-4u+4)\cdot u^{\frac{1}{2}}\,du$$
$$=\int u^{2+\frac{1}{2}}-4\cdot u^{1+\frac{1}{2}}+4\cdot u^{\frac{1}{2}}\,du$$
$=$ (not finished yet)

48. $\int x^3\sqrt{x^2+1}\,dx \xrightarrow[du=2x\,dx]{u=x^2+1\Rightarrow x^2=u-1} \int x^2\cdot x\cdot\sqrt{x^2+1}\,dx$

$$=\int x^2\cdot\sqrt{x^2+1}\cdot x\,dx$$
$$=\int (u-1)\cdot\sqrt{u}\cdot\frac{du}{2}$$ = (to be completed)

67. $\int_1^2 x\sqrt{x-1}\,dx \xrightarrow[du=dx]{u=x-1\Rightarrow x=u+1} \int_{1-1}^{2-1}(u+1)\cdot\sqrt{u}\cdot du=\int_0^1(u\sqrt{u}+\sqrt{u})\,du$

= (to be completed)